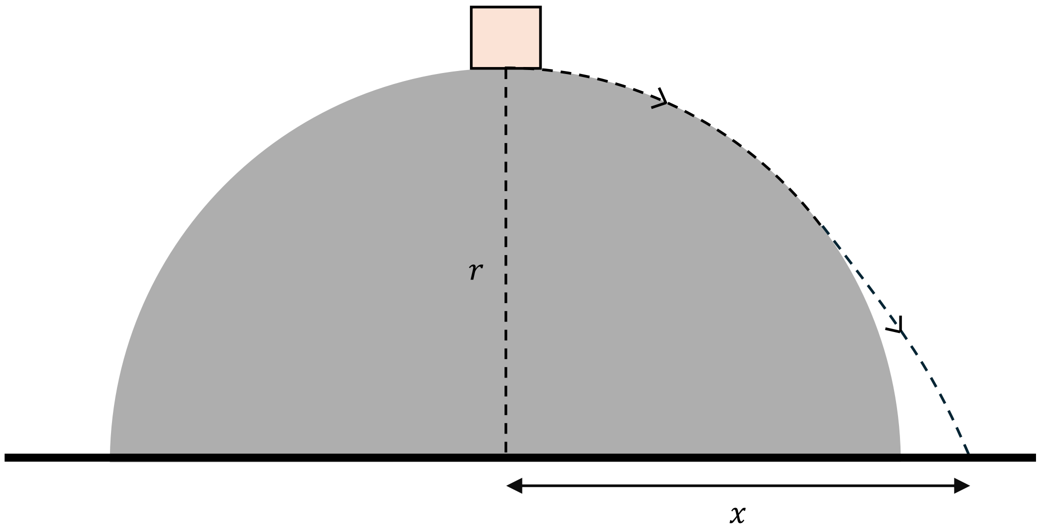
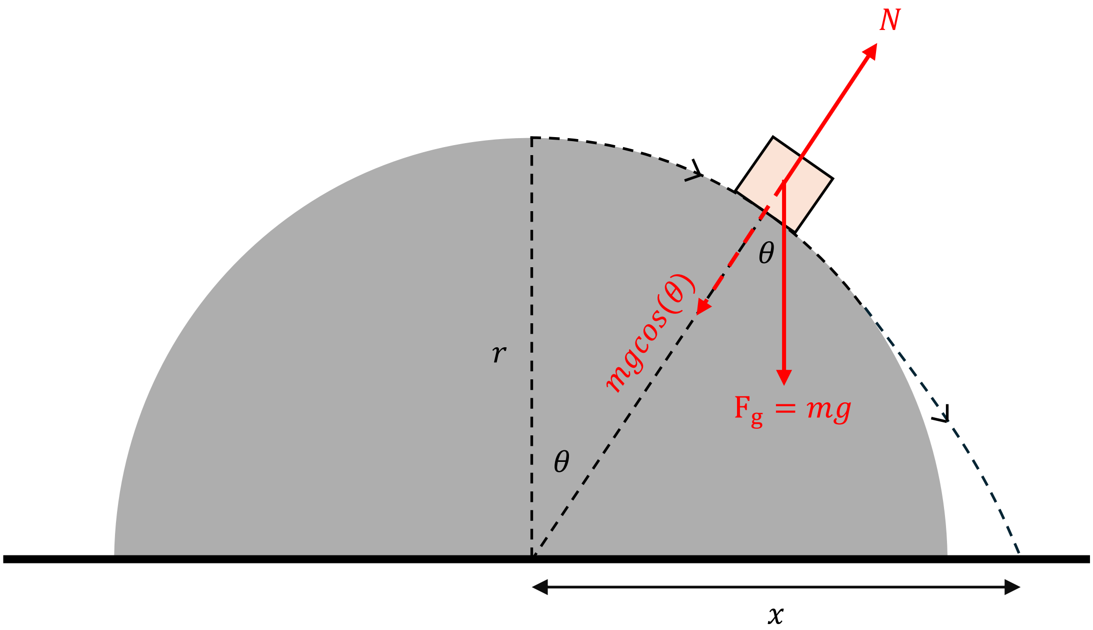
Solution 1: Starting from rest, a block falls from a frictionless semi-circle. Find x.



While block remains on semi-circle. The radial component of the gravitational force (F_g) minus the normal force (N) must equal the centripetal force (F_c).

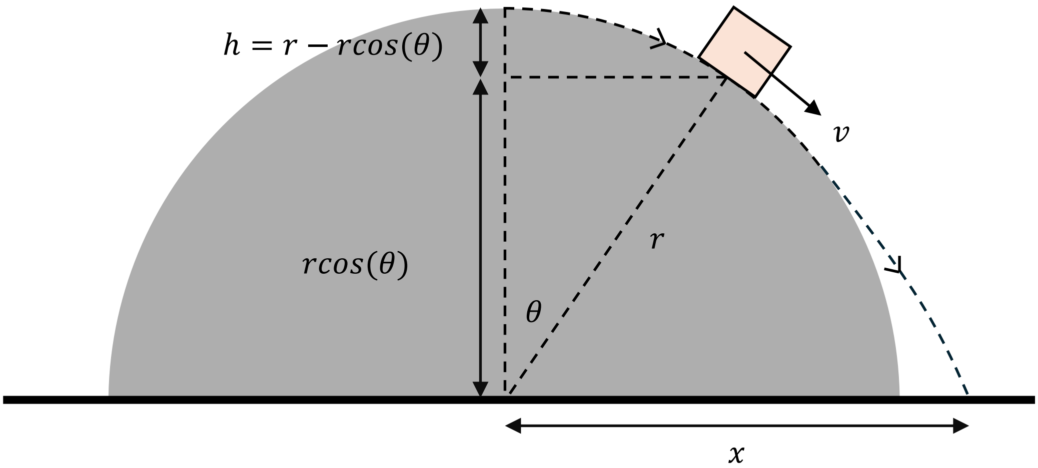


$$N = mgcos(\theta) - F_c = mgcos(\theta) - \frac{mv^2}{r}$$

Where v is the radial velocity of the block. When the normal force is 0, the block no longer remains in contact with the semi-circle.

$$mgcos(\theta) - \frac{mv^2}{r} = 0, \qquad gcos(\theta) = \frac{v^2}{r}, \qquad v^2 = grcos(\theta)$$

Now apply conservation of energy to obtain another equation of v and θ.



$$mgh = \frac{1}{2}mv^2$$

$$mg\big(r - rcos(\theta)\big) = \frac{1}{2}mv^2$$

$$2gr(1 - cos(\theta)) = v^2$$

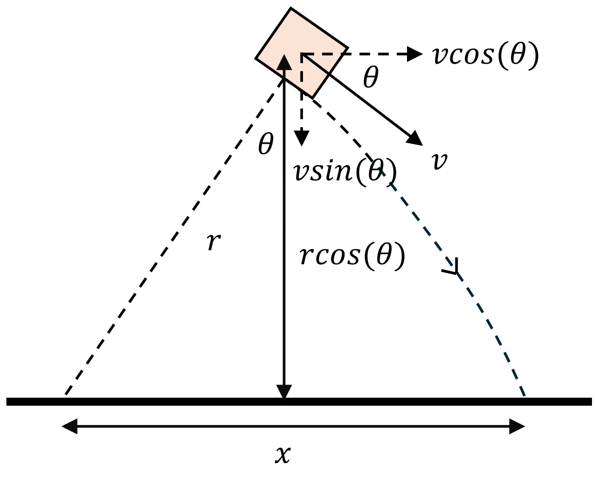
Solving for v and θ:

$$v^2 = grcos(\theta) = 2gr(1 - cos(\theta))$$

$$cos(\theta) = 2(1 - cos(\theta))$$

$$cos(\theta) = \frac{2}{3}, \qquad sin(\theta) = \sqrt{1 - \frac{2^2}{3^2}} = \frac{\sqrt{5}}{3}, \qquad v^2 = \frac{2}{3}gr$$

After this point, the normal force is 0 and the block enters a parabolic trajectory.



$$y = y_0 - v_{0y}t - \frac{1}{2}gt^2, \qquad x = x_0 + v_{0x}t$$

When $y = 0$:

$$y_0 - v_{0y}t - \frac{1}{2}gt^2 = 0, \qquad t = -\frac{v_{0y} \pm \sqrt{v_{0y}^2 + 4\frac{1}{2}gy_0}}{2\frac{1}{2}g} = -\frac{v_{0y}}{g} \pm \sqrt{\frac{v_{0y}^2}{g^2} + \frac{2y_0}{g}}$$

Taking the positive solution:

$$t = -\frac{v_{0y}}{g} + \sqrt{\frac{v_{0y}^2}{g^2} + \frac{2y_0}{g}} = -\frac{vsin(\theta)}{g} + \sqrt{\frac{v^2\sin^2(\theta)}{g^2} + \frac{2rcos(\theta)}{g}}$$

$$t = -\frac{\sqrt{\frac{2}{3}gr}\frac{\sqrt{5}}{3}}{g} + \sqrt{\frac{\frac{2}{3}gr\left(\frac{\sqrt{5}}{3}\right)^2}{g^2} + \frac{2r\frac{2}{3}}{g}}$$

$$t = \sqrt{\frac{r}{g}}\left(-\sqrt{\frac{2}{3}}\frac{\sqrt{5}}{3} + \sqrt{\frac{2}{3}\left(\frac{\sqrt{5}}{3}\right)^2 + 2\frac{2}{3}}\right)$$

$$t = \frac{1}{\sqrt{27}}\sqrt{\frac{r}{g}}\left(-\sqrt{10} + \sqrt{46}\right)$$

Finding x:

$$x = x_0 + v_{0x}t = rsin(\theta) + vcos(\theta)t$$

$$= r\frac{\sqrt{5}}{3} + \sqrt{\frac{2}{3}gr}\frac{2}{3}\frac{1}{\sqrt{27}}\sqrt{\frac{r}{g}}\left(-\sqrt{10} + \sqrt{46}\right)$$

$$= r\frac{1}{27}\left(9\sqrt{5} + \sqrt{8}\left(-\sqrt{10} + \sqrt{46}\right)\right)$$

$$\boldsymbol{x = r\frac{1}{27}\left(5\sqrt{5} + 4\sqrt{23}\right)}$$

Plot of trajectory for $r = 1$:

```
θ = ArcCos[2 / 3];
v = √(2/3) √(g r);
x = 1/27 (5 √5 + 4 √23) r;
r = 1;
g = 9.81;

Show[
 PolarPlot[r, {x, Pi / 2 - θ, Pi / 2}, PlotStyle → Red],
 ParametricPlot[{v Cos[θ] t + r Sin[θ], -v Sin[θ] t - 1 / 2 g t ^ 2 + r Cos[θ]}, {t, 0, t0}, PlotStyle → Red],
 RegionPlot[x ^ 2 + y ^ 2 < r ^ 2, {x, -1.2 r, 1.2 r}, {y, 0, 1.2 r}, BoundaryStyle → None]
]
```